MOTION OF A LIQUID FILM ON THE SURFACE OF A ROTATING CYLINDER IN A GRAVITATIONAL FIELD

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1. Statement of the Problem

We consider the two-dimensional motion of a liquid film on the surface of a rotating cylinder in a gravitational field (Fig. 1); a is the radius of the cylinder, ω is its constant angular velocity about its axis, g is the acceleration due to gravity, ν is the kinematic viscosity, ρ is its density, and σ is the surface tension. Three independent dimensionless combinations can be formed from these quantities: the Reynolds number $\mathrm{Re} = a^2\omega/\nu$, the Galileo number $\gamma = g/a\omega^2$, and the inverse Weber number $\beta = \sigma/\rho a^3\omega^2$.

The problem consists in determining the function $h(\theta, t)$ positive for $t \in [0, T]$ and all θ, and the solution u, v, p of the Navier–Stokes equations

$$u_t + uu_r + \frac{v}{r} u_\theta - \frac{v^2}{r} = -p_r + \mathrm{Re}^{-1}\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} - \frac{2}{r^2} v_\theta - \frac{u}{r^2}\right) - \gamma \sin\theta, \tag{1.1}$$

$$v_t + uv_r + \frac{v}{r} v_\theta + \frac{uv}{r} = -\frac{1}{r} p_\theta + \mathrm{Re}^{-1}\left(v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} + \frac{2}{r^2} u_\theta - \frac{v}{r^2}\right) - \gamma \cos\theta,$$

$$(ru)_r + v_\theta = 0,$$

in the region $1 < r < 1 + h(\theta, t)$, $|\theta| < \infty$, $0 < t < T$ which satisfy the boundary conditions

$$u = 0,\ v = 1 \quad \text{at} \quad r = 1; \tag{1.2}$$

$$h_t + \frac{v}{r} h_\theta - u = 0 \quad \text{at} \quad r = 1 + h; \tag{1.3}$$

$$\left(1 - \frac{1}{r^2} h_\theta^2\right)\left(v_r - \frac{v}{r} + \frac{1}{r} u_\theta\right) + \frac{2}{r} h_\theta \left(u_r - \frac{u}{r} - \frac{1}{r} v_\theta\right) = 0 \quad \text{at} \quad r = 1 + h; \tag{1.4}$$

$$\beta\left[(1+h)^2 + 2h_\theta^2 - (1+h)h_{\theta\theta}\right]\left[(1+h)^2 + h_\theta^2\right]^{-3/2} = p - 2\,\mathrm{Re}^{-1}\left(1 + \frac{1}{r^2} h_\theta^2\right)^{-1}\left[u_r - \frac{1}{r} h_\theta \left(v_r - \frac{v}{r} + \frac{1}{r} u_\theta\right) + \frac{1}{r^2} h_\theta^2 \left(\frac{1}{r} v_\theta + \frac{u}{r}\right)\right] \quad \text{at} \quad r = 1 + h, \tag{1.5}$$

the condition that h, u, v, and p be periodic in θ with the period 2π, and the initial conditions

$$h = h_0(\theta),\ u = u_0(r, \theta),\ v = v_0(r, \theta) \quad \text{at} \quad t = 0. \tag{1.6}$$

Here h_0, u_0, and v_0 are given functions which are periodic in θ with the period 2π, and $(ru_0)_r + v_{0,\theta} = 0$.

All quantities in Eqs. (1.1)-(1.6) are dimensionless. We take the expressions a, $1/\omega$, $a\omega$, and $\rho a^2\omega^2$ as length, time, velocity, and pressure scales, respectively. The Navier–Stokes equations (1.1) are written in polar coordinates r, θ with the pole at the center of a cross section of the cylinder and the polar axis perpendicular to the direction of gravity; u and v are the radial and peripheral velocity components.

Conditions (1.2) are the conditions for no slipping between the liquid and the surface of the rotating cylinder. Equation (1.3) states that the free boundary $r = 1 + h(\theta, t)$ bounds the liquid volume. Equation (1.4) expresses the absence of tangential stress at the free boundary and (1.5), the equality of the normal stress and the capillary pressure.

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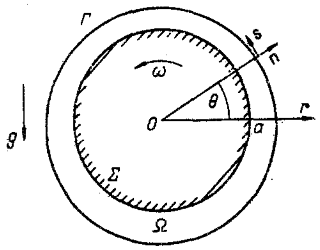

Fig. 1

A rigorous mathematical treatment of the unsteady motion of a viscous liquid with a free boundary, a class of problems which includes (1.1)-(1.6), was begun relatively recently. By neglecting surface tension and assuming that the entire boundary of the liquid is a free surface, Solonnikov [1] proved that the Navier–Stokes equations for the unsteady problem have a unique solution on a certain finite time interval. By making a small change in the treatment in [1] this theorem can be extended to the case of a boundary of the flow region consisting of two nonintersecting parts: a free boundary and a smooth solid surface on which the nonslip condition is satisfied. This leads to the following statement on the solvability of problem (1.1)-(1.6) when $\beta = 0$.

We assume that the function $h_0(\theta)$ is positive, periodic in θ with the period 2π, and belongs to the Hölder class $C^{2+\alpha}$, $0 < \alpha < 1$, and that the functions $u_0(r, \theta)$, $V_0(r, \theta)$ are also periodic in θ with the period 2π and belong to the class $C^{2+\alpha}$ in the closed region $\overline{\Omega_0}$: $1 \leq r \leq 1 + h_0(\theta)$. In addition, we assume that the initial velocity field u_0, v_0 satisfies the equation of continuity in the region Ω_0, the nonslip condition (1.2) at $r = 1$, and the condition for no tangential stress (1.4) at $r = 1 + h_0$. Then there exists a $T > 0$ such that problem (1.1)-(1.6) with $\beta = 0$ has a unique solution on the interval $0 \leq t \leq T$.

The question of the solvability (even local) of problem (1.1)-(1.6) when $\beta \neq 0$ so far remains open.

The problem under consideration has a number of technical applications [2]. In these applications the thickness of the liquid film is generally small in comparison with the radius of the cylinder. The approximate equations describing the motion of a thin film on the surface of a rotating cylinder for small Reynolds numbers were derived in [2].

The problem of the steady flow of a heavy liquid on the surface of a rotating cylinder is of independent interest. It consists in finding the function $h(\theta)$ and the time-independent solution of the system (1.1) which satisfies conditions (1.2)-(1.5) and one of the following subsidiary conditions:

$$\int_1^{1+h(\theta)} v(r, \theta)\, dr = q; \tag{1.7}$$

$$\int_0^{2\pi} h(\theta)\, d\theta = 2\pi \bar{h} \tag{1.8}$$

with given positive constants q and $\bar{h}$. Condition (1.8) gives the average thickness of the film and (1.7), the flow rate of liquid through a cross section of the film. It is clear that the integral on the left-hand side of Eq. (1.7) does not depend on θ. Condition (1.7) or (1.8) is necessary to separate the unique solution from the one-parameter family of steady solutions of problem (1.1)-(1.5).

A theorem on the existence and uniqueness of the solution of the steady problem (1.1)-(1.5), (1.7) for small Galileo numbers γ was announced in [3].

2. A Theorem on the Existence and Uniqueness of the Steady Solution

We consider system (1.1) in the steady case. Introducing the stream function $\psi(r, \theta)$ by the relations

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v = \frac{\partial\psi}{\partial r}, \tag{2.1}$$

we replace system (1.1) by a single equation for the stream function

$$\Delta\Delta\psi - \frac{R}{r}\frac{\partial(\psi, \Delta\psi)}{\partial(r, \theta)} = 0, \tag{2.2}$$

where $\Delta\psi = r^{-1}(r\psi_r)_r + r^{-2}\psi_{\theta\theta}$.

In the steady case we denote the region occupied by the liquid by Ω and the free boundary, by Γ. The nonslip conditions (1.2) are satisfied on the outer boundary Σ of region Ω as a consequence of (2.1) and can be written in the form

$$\psi = 0, \ \partial\psi/\partial r = 1 \ \text{at} \ r = 1. \tag{2.3}$$

We fix the arbitrariness in the determination of the stream function by setting $\psi = 0$ on the streamline $r = 1$. In the steady case the kinematic condition on the free boundary (1.3) and the flow rate condition (1.7) lead to the relation

$$\psi = q \ \text{at} \ r = 1 + h(\theta). \tag{2.4}$$

The dynamic conditions (1.4) and (1.5) lead to the following conditions for the stream function on the curve Γ [3]:

$$\Delta\psi - 2K\partial\psi/\partial n = 0 \quad \text{at} \quad r = 1 + h(\theta); \tag{2.5}$$

$$\frac{\partial\Delta\psi}{\partial n} + 2\frac{\partial^3\psi}{\partial s^2\partial n} - \frac{\mathrm{Re}}{2}\frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial n}\right)^2 + \beta\frac{\partial K}{\partial s} - \frac{\gamma\left[(1+h)\cos\theta + h_\theta\sin\theta\right]}{\left[(1+h)^2 + h_\theta^2\right]^{1/2}} = 0 \quad \text{at} \quad r = 1 + h(\theta), \tag{2.6}$$

where K is the curvature of the curve Γ, with $K > 0$ if Γ is convex outward from the liquid. The expression $\partial/\partial s$ denotes differentiation along the tangent and $\partial/\partial n$, along the outward normal to Γ. The positive direction of the tangent to the curve Γ is chosen so that the tangential component of the vector **s** is positive (cf. Fig. 1). Condition (2.5) follows from (1.4), (1.3), and (2.1). To obtain (2.6) it is necessary to differentiate Eq. (1.5) along Γ and to use (1.1) to eliminate the term $\partial p/\partial s$ which arises. To the conditions enumerated is added the periodicity condition

$$\psi(r, \theta + 2\pi) \equiv \psi(r, \theta), \ h(\theta + 2\pi) \equiv h(\theta). \tag{2.7}$$

The final formulation of the steady problem is the following: to determine the positive function $h(\theta)$ and the solution $\psi(r, \theta)$ of Eq. (2.2) in the region $1 < r < 1 + h(\theta)$ so that Eqs. (2.3)-(2.7) are satisfied. If $\gamma = 0$ (no gravity), the problem has a trivial solution $\psi = (r^2 - 1)/2$, $h = \sqrt{1 + 2q} - 1 \equiv b = \text{const}$, describing the rotation of the liquid as a rigid body. It can be shown that this solution is unique. Solutions of problem (2.2)-(2.7) which are close to trivial are analyzed below.

We study this problem with an unknown boundary by transforming to new independent variables θ, ψ (von Mises variables) and a new unknown function $r = \zeta(\theta, \psi)$. According to (2.3) and (2.4), in the θ, ψ plane the surface of the cylinder corresponds to the straight line $\psi = 0$ and the free boundary, to the straight line $\psi = q$. As a consequence of (2.2) the function ζ satisfies the equation

$$\zeta_\psi A\left(A\zeta - \frac{1}{\zeta\zeta_\psi}\right) - \frac{1}{2}\left[\left(A\zeta - \frac{1}{\zeta\zeta_\psi}\right)^2\right]_\psi + \frac{\mathrm{Re}}{\zeta\zeta_\psi}\left(A\zeta - \frac{1}{\zeta\zeta_\psi}\right)_\theta = 0, \tag{2.8}$$

where A denotes the differential operator

$$A = \frac{1}{\zeta^2\zeta_\psi}\frac{\partial^2}{\partial\theta^2} - \frac{2\zeta_\theta}{\zeta^2\zeta_\psi^2}\frac{\partial^2}{\partial\theta\partial\psi} + \frac{\zeta^2 + \zeta_\theta^2}{\zeta^2\zeta_\psi^2}\frac{\partial^2}{\partial\psi^2}.$$

The nonslip conditions (2.3) lead to the following conditions for the function ζ:

$$\zeta - 1 = 0, \ \zeta_\psi - 1 = 0 \ \text{for} \ \psi = 0. \tag{2.9}$$

From conditions (2.5) and (2.6) at the free boundary it follows that

$$A\zeta - \frac{1}{\zeta\zeta_\psi} + \frac{2\left(\zeta^2 + 2\zeta_\theta^2 - \zeta\zeta_{\theta\theta}\right)}{\zeta\zeta_\psi\left(\zeta^2 + \zeta_\theta^2\right)} = 0 \quad \text{for} \quad \psi = q; \tag{2.10}$$

$$-\frac{\sqrt{\zeta^2 + \zeta_\theta^2}}{\zeta\zeta_\psi}\left(A\zeta - \frac{1}{\zeta\zeta_\psi}\right)_\psi + \frac{2}{\sqrt{\zeta^2 + \zeta_\theta^2}}\left[\frac{1}{\sqrt{\zeta^2 + \zeta_\theta^2}}\left(\frac{\sqrt{\zeta^2 + \zeta_\theta^2}}{\zeta\zeta_\psi}\right)_\theta\right]_\theta +$$
$$+ \frac{\zeta_\theta}{\zeta\sqrt{\zeta^2 + \zeta_\theta^2}}\left(A\zeta - \frac{1}{\zeta\zeta_\psi}\right)_\theta - \frac{R}{2\sqrt{\zeta^2 + \zeta_\theta^2}}\left(\frac{\zeta^2 + \zeta_\theta^2}{\zeta^2\zeta_\psi^2}\right)_\theta +$$
$$+ \frac{\beta}{\sqrt{\zeta^2 + \zeta_\theta^2}}\left[\frac{\zeta^2 + 2\zeta_\theta^2 - \zeta\zeta_{\theta\theta}}{\sqrt{(\zeta^2 + \zeta_\theta^2)^3}}\right]_\theta - \frac{\gamma\left(\zeta\cos\theta + \zeta_\theta\sin\theta\right)}{\sqrt{\zeta^2 + \zeta_\theta^2}} = 0 \quad \text{for} \quad \psi = q. \tag{2.11}$$

It is required to find the solution of Eq. (2.8) in the strip $\Pi: 0 < \psi < 1$, which satisfies conditions (2.9)-(2.11) and is periodic in θ with the period 2π.

We denote by $C_p^{m+\alpha}(\bar{\Pi})$ the subspace of the Hölder space $C^{m+\alpha}(\bar{\Pi})$ formed by functions which are periodic in θ with the period 2π. Here $m \geq 0$ is an integer, $0 < \alpha < 1$, and $\bar{\Pi}$ is the closure of Π. We denote by $|\ .\ |_{\Pi}^{(m+\alpha)}$ the norm in this space. The symbols $C_p^{m+\alpha}(E)$, $|\ .\ |_E^{(m+\alpha)}$ have a similar meaning, where E is a real straight line. $C_0, C_1, \ldots$ are positive constants. We define a function $\zeta_0 = \sqrt{1+2\psi}$. This function is the solution of problem (2.8)-(2.11) for $\gamma = 0$.

THEOREM. For all $q > 0$, $\mathrm{Re} \geq 0$, and $\beta \geq 0$ there exists a $\gamma_0 > 0$ such that for $\gamma \in [0, \gamma_0]$ problems (2.8)-(2.11) has the solution $\zeta \in C_p^{4+\alpha}(\bar{\Pi})$, which is unique in a certain sphere $|\zeta - \zeta_0|_{\Pi}^{(4+\alpha)} \leqslant C_0$.

We prove this theorem by Newton's method. We define a Banach space X as the space of the vector functions $\mathbf{x} = \{x_1, \ldots, x_5\}$, where $x_1(\theta, \psi) \in C_p^{\alpha}(\bar{\Pi})$, $x_k(\theta) \in C_p^{6-k+\alpha}(E)$ for $k = 2, \ldots, 5$. The norm in this space is

$$\|\mathbf{x}\|_X = |x_1|_{\Pi}^{(\alpha)} + \sum_{k=2}^{5} |x_k|_E^{(6-k+\alpha)}.$$

We treat the boundary value problem (2.8)-(2.11) as an operator equation of the form

$$\mathbf{F}(\zeta) = 0, \tag{2.12}$$

where the operator $\mathbf{F}(\zeta)$ is defined in a sphere $B_c: |\zeta - \zeta_0|_{\Pi}^{(4+\alpha)} \leqslant c$ of the space $C_p^{4+\alpha}(\bar{\Pi}) = Z$ by the equation $\mathbf{F}(\zeta) = \{F_1(\zeta), \ldots, F_5(\zeta)\}$ and operates in the space X. The components $F_1, \ldots, F_5$ of the vector $\mathbf{F}$ are the differential expressions on the left-hand sides of Eq. (2.8), the first and second conditions (2.10) and (2.11), respectively. For example, $F_3(\zeta) = \zeta_\psi - 1$ for $\psi = 0$.

We take the Fréchet derivative of the operator $\mathbf{F}(\zeta)$ in the sphere $B_c (0 < c < 1)$. We denote by $\mathbf{F}'_z(\zeta)$ its Fréchet derivative at the point $z \in B_c$ and set $\|\zeta\|_Z = |\zeta|_{\Pi}^{(4+\alpha)}$. Simple calculations show that for all ζ_1 and ζ_2 from B_c

$$\|\mathbf{F}'_{\zeta_1} - \mathbf{F}'_{\zeta_2}\|_X \leqslant (C_3 + \gamma C_4)\|\zeta_1 - \zeta_2\|_Z,$$

where C_3 and C_4 depend only on Re, β, α, and c. It is clear also that $\mathbf{F}(\zeta_0) = \{0, 0, 0, 0, -\gamma\cos\theta\}$ and $\|\mathbf{F}(\zeta_0)\|_X \leq 4\gamma$. For the following it is essential that the linear operator $\mathbf{F}'_{\zeta_0}$ have an inverse $(\mathbf{F}'_{\zeta_0})^{-1}: X \to Z$, and that the estimate (proved later)

$$\|(\mathbf{F}'_{\zeta_0})^{-1}\| \leqslant C_1 \tag{2.13}$$

be valid for all fixed $q > 0$, $\mathrm{Re} \geqslant 0$, $\beta \geqslant 0$, $\alpha \in (0, 1)$ and all γ from the interval $[0, \gamma_1]$ if $\gamma_1 > 0$ is sufficiently small.

We introduce the notation $C_2 = C_3 + \gamma_0 C_4$ and choose γ_0 $(0 < \gamma_0 \leq \gamma_1)$ small enough so that $16\gamma_0 C_1^2 C_2 < 1$. Then for $\gamma \in [0, \gamma_0]$ the conditions of Kantorovich's theorem on the convergence of Newton's operator method [4] are satisfied for Eq. (2.12). According to this theorem, if $\delta = 16\gamma_0 C_1^2 C_2 < 1$, Eq. (2.12) for $0 \leq \gamma \leq \gamma_0$ has a unique solution in the sphere $\|\zeta - \zeta_0\|_Z \leq C_0$, where $C_0 = \min[C_1(1 - \sqrt{1-\delta})/2C_2, c]$. The sequence $\{\zeta_n\}$, in which $\zeta_0 = \sqrt{1+2\psi}$ and

$$\zeta_n = \zeta_{n-1} - (\mathbf{F}'_{\zeta_0})^{-1}\mathbf{F}(\zeta_{n-1}) \tag{2.14}$$

for $n \geq 1$ converges to this solution.

It remains to prove that the operator $\mathbf{F}'_{\zeta_0}$ has an inverse and to derive the estimate (2.13). Let us consider the linear equation

$$\mathbf{F}'_{\zeta_0}(\zeta) = \mathbf{x}, \tag{2.15}$$

where $\mathbf{x}$ is an arbitrary element of the space X. The change of variables $r = \sqrt{1+2\psi}$, $\zeta = \varphi/r$ transforms the homogeneous problem for $\varphi(r, \theta)$ corresponding to (2.15) to the form

$$\Delta\Delta\varphi - \mathrm{Re}\,\Delta\varphi_\theta = 0 \quad \text{for} \quad 1 < r < 1 + b = \sqrt{1+2q}, \tag{2.16}$$

$$\varphi = \varphi_r = 0 \quad \text{at} \quad r = 1,$$

$$\Delta\varphi - \frac{2}{r^2}\varphi_{\theta\theta} - \frac{2}{r}\varphi_r = 0 \quad \text{at} \quad r = 1 + b,$$

$$(\Delta\varphi)_r + \frac{2}{r^2}\left(\varphi_r - \frac{\varphi}{r}\right)_{\theta\theta} - \mathrm{Re}\left(\varphi_r - \frac{\varphi}{r}\right) + \frac{\beta}{r^4}(\varphi_{\theta\theta} + \varphi)_\theta +$$

$$+\frac{\gamma}{r^2}(\varphi_\theta \sin\theta + \varphi\cos\theta) = 0 \quad \text{at} \quad r = 1 + b,$$

$$\varphi(r, \theta + 2\pi) \equiv \varphi(r, \theta).$$

We show that for sufficiently small values of γ this problem has only a trivial solution. The solution of problem (2.16) satisfies the identity

$$\int_1^{1+b}\int_0^{2\pi}\left[\left(\varphi_{rr} - \frac{1}{r}\varphi_r - \frac{1}{r^2}\varphi_{\theta\theta}\right)^2 + 4\left(\frac{1}{r}\varphi_{r\theta} - \frac{1}{r^2}\varphi_\theta\right)^2\right] r dr d\theta = -\frac{\gamma}{4(1+b)^2}\int_0^{2\pi}\varphi^2(1+b, \theta)\cos\theta d\theta, \tag{2.17}$$

which is obtained by multiplying the first of Eqs. (2.16) by φ, integrating over the annulus $1 < r < 1 + b$, and using the boundary conditions. By virtue of Korn's inequality for two-dimensional solenoidal vector fields [5] and a theorem of Sobolev on traces, it follows from (2.17) that $\varphi = 0$ if $0 \le \gamma \le \gamma_1$ and γ_1 is sufficiently small.

The boundary operators in (2.16) satisfy the complementarity condition [6] with respect to the operator $\Delta\Delta - \mathrm{Re}\Delta\partial/\partial\theta$. Therefore, the absence of nontrivial solutions of the homogeneous problem (2.16) implies the unique solvability of the corresponding inhomogeneous problem and Eq. (2.15) for any $x \in X$. The inequality (2.13) follows from a priori estimates in [6]. Thus the theorem is proved.

We note that as $\gamma \to 0$ the solution of problem (2.1)-(2.11) can be estimated by

$$|\zeta - \sqrt{1 + 2\psi}|_{\Pi}^{(4+\alpha)} = O(\gamma). \tag{2.18}$$

For sufficiently small γ this estimate ensures the one-sheeted mapping of the strip Π onto the flow region $1 < r < 1 + h(\theta)$, $\theta \in E$. The solution $\psi \in C^{4+\alpha}(\bar{\Omega})$, $h \in C^{4+\alpha}(E)$ of the two-dimensional steady problem of a film corresponds to the solution $\zeta \in C^{4+\alpha}(\bar{\Pi})$ of problem (2.8)-(2.11).

The asymptotic solution of problem (2.8)-(2.11) for small γ can be refined. According to (2.14) the second term ζ_1 of the recurrent sequence $\{\zeta_n\}$ as $\gamma \to 0$ can be written

$$\zeta_1 = \sqrt{1 + 2\psi}\,[1 + \gamma(1 + 2\psi)^{-1}\varphi(\sqrt{1 + 2\psi}, \theta)] + O(\gamma^2).$$

Here $\varphi(r, \theta)$ is the solution of the linear problem (2.16) in which the second boundary condition at $r = 1 + b$ is replaced by

$$(\Delta\varphi)_r + \frac{2}{r^2}\left(\varphi_r - \frac{\varphi}{r}\right)_{\theta\theta} - \mathrm{Re}\left(\varphi_r - \frac{\varphi}{r}\right)_\theta + \frac{\beta}{r^4}(\varphi_{\theta\theta} + \varphi)_\theta = -\cos\theta \quad \text{at} \quad r = 1 + b.$$

The following estimate holds:

$$|\zeta - \zeta_1|_{\Pi}^{(4+\alpha)} = O(\gamma^2) \quad \text{as} \quad \gamma \to 0. \tag{2.19}$$

To derive Eqs. (2.18) and (2.19) it is sufficient to use the estimate of the error of the modified Newton's method [4]. Equation (2.19) estimates how closely the linear term ζ_1 approximates the solution ζ of problem (2.8)-(2.11) for small γ.

Remark 1. The solution of problem (2.8)-(2.11), which means also the solution of the initial problem (2.2)-(2.7), depends continuously on the parameter β in any finite range of its variation. In particular, it is assumed that $\beta = 0$ (zero surface tension).

Remark 2. Using the estimates of the solutions of elliptic equations [6] it can be shown that the solution $\zeta(\psi, \theta) \in C_p^{4+\alpha}(\bar{\Pi})$ of problem (2.8)-(2.11) actually belongs to the class $C^\infty(\bar{\Pi})$. This implies that the velocity field and the free boundary corresponding to it are also infinitely differentiable.

Remark 3. In formulating the two-dimensional steady problem of a film it would be possible to specify the average film thickness $\bar{h}$ rather than the flow rate q. In view of the relation $h(\theta) = \zeta(\theta, q) - 1$ and (2.18) there is a one-to-one relation between these quantities for small γ

$$\bar{h} = \sqrt{1 + 2q} - 1 + O(\gamma).$$

3. Equations of Motion of a Thin Film

Let us assume that the initial value of the function h has the form $h_0 = \varepsilon H_0(\theta)$, where $\varepsilon > 0$ is a small parameter. The parameter ε has the meaning of the ratio of a characteristic (for example, average) thickness of the film to the radius of the cylinder. The starting point for the derivation of the equations of a thin film is the representation

$$h = \varepsilon H, \; r = 1 + \varepsilon y, \; u = \varepsilon U, \; v = V, \tag{3.1}$$

and the assumption that H, U, V and their derivatives with respect to t, θ, and the new independent variable y remain finite as $\varepsilon \to 0$. Substituting (3.1) into the first two equations of (1.1) and condition (1.5) leads to the equations

$$-V^2 + O(\varepsilon) = -\frac{1}{\varepsilon} P_y + \frac{1}{\mathrm{Re}\beta}[U_{yy} + O(\varepsilon)] - \gamma \sin\theta, \tag{3.2}$$

$$V_t + UV_y + VV_\theta + O(\varepsilon) = -[1 + O(\varepsilon)] P_\theta +$$

$$+ \frac{1}{\mathrm{Re}\varepsilon^2}[V_{yy} + O(\varepsilon)] - \gamma\cos\theta \quad \text{for} \quad 0 < y < H(\theta, t);$$

$$-\beta\varepsilon[H_{\theta\theta} + H + O(\varepsilon)] = P - 2\mathrm{Re}^{-1}[U_y - O(\varepsilon)] \quad \text{for } y = H(\theta, t). \tag{3.3}$$

where $p = {}_p\beta$. In deriving Eq. (3.3) account was taken of the fact that in view of (1.4) $V_y(H, \theta, t) = O(\varepsilon)$ as $\varepsilon \to 0$.

We determine $P(y, \theta, t)$ by integrating the first of Eqs. (3.2) with respect to y, using (3.3), and substituting the result into the second of Eqs. (3.2). We obtain the relation

$$\mathrm{Re}\varepsilon^2[V_t + UV_y + VV_\theta + O(\varepsilon)] = V_{yy} + O(\varepsilon) +$$

$$+ \beta\mathrm{Re}\varepsilon^3[H_{\theta\theta\theta} + H_\theta + O(\varepsilon)] - \gamma\mathrm{Re}\varepsilon^2[\cos\theta + O(\varepsilon)] \tag{3.4}$$

as $\varepsilon \to 0$. We introduce the notation

$$\mathrm{Re}\varepsilon^2 = \varkappa, \ \beta\mathrm{Re}\varepsilon^3 = \chi, \ \gamma\mathrm{Re}\varepsilon^2 = \mu.$$

We postulate that in the limit $\varepsilon \to 0$ the quantities $\varkappa$, χ, and μ approach finite values. Physically this assumption means that in Eq. (3.4) the inertial, viscous, capillary, and gravitational forces are of the order of magnitude for small ε. Letting $\varepsilon \to 0$ in Eq. (3.4) we obtain the equation

$$\varkappa(V_t + UV_y + VV_\theta) = V_{yy} - \mu\cos\theta + \chi(H_{\theta\theta} + H_\theta). \tag{3.5}$$

Substituting (3.1) into the last of Eqs. (1.1) and into (1.2)-(1.4) and letting $\varepsilon \to 0$ we obtain

$$U_y + V_\theta = 0; \tag{3.6}$$

$$U = 0, \ V = 1 \quad \text{at} \quad y = 0; \tag{3.7}$$

$$H_t + VH_\theta - U = 0 \ \text{for} \quad y = H; \tag{3.8}$$

$$V_y = 0 \ \text{for} \quad y = H. \tag{3.9}$$

We note that the relation of the orders of magnitude of u and v indicated in (3.1) as $\varepsilon \to 0$ is unique, and the limiting equation of continuity (3.6) is nontrivial.

To the boundary conditions (3.7)-(3.9) for system (3.5), (3.6) must be added the condition that U, V, and H must be periodic in θ with the period 2π, and the initial conditions

$$H = H_0(\theta) \quad \text{at} \quad t = 0; \tag{3.10}$$

$$V = V_0(y, \theta) \quad \text{at} \quad t = 0, \tag{3.11}$$

where H_0 and V_0 are given functions. The latter is defined in the domain $0 < y < H_0(\theta)$. As a consequence of (3.6) and (3.7) the specification of V at t = 0 uniquely determines the initial value of U. We note that for the initial condition (1.6) for v to agree with the representation (3.1) it is necessary to assume that $V_0 = \lim v_0(1 + \varepsilon y, \theta)$ as $\varepsilon \to 0$.

Equations (3.5)-(3.11) constitute a boundary-value problem whose solution is interpreted as the motion in a thin layer of liquid on the surface of a rotating cylinder. Problem (3.5)-(3.11) is rather complicated in view of its nonlinearity, the degeneracy of Eqs. (3.5) and (3.6), and the presence of an unknown boundary. We restrict ourselves to the construction of the formally asymptotic solution of this problem for $\varkappa \to 0$. The parameters χ and μ can take on any nonnegative values including zero.

We set $\varkappa = 0$ in Eq. (3.5). Integrating the equation obtained twice with respect to y and using (3.7) and (3.9) we find

$$V = (y^2/2 - yH)[\mu\cos\theta - \chi(H_{\theta\theta} + H)_\theta] + 1. \tag{3.12}$$

From (3.6), (3.7), and (3.12) the expression for U is

$$U = \frac{y^2}{2}\left(H - \frac{y}{3}\right)[\mu\cos\theta - \chi(H_{\theta\theta} + H)_\theta]_\theta + \frac{y^2}{2} H_\theta[\mu\cos\theta - \chi(H_{\theta\theta} + H)_\theta]. \tag{3.13}$$

Substituting Eqs. (3.12) and (3.13) into (3.8) we find the differential equation for $H(\theta, t)$

$$H_t + \left(H - \frac{\mu}{3} H^3 \cos\theta\right)_\theta + \frac{\chi}{3}\left[H^3 (H_{\theta\theta\theta} + H_\theta)\right]_\theta = 0. \tag{3.14}$$

Equation (3.10) gives the initial condition for this equation.

Suppose $\chi > 0$. The linearization of Eq. (3.14) for any positive solution leads to a parabolic equation according to Petrovskii [7]. By using a priori estimates of solutions of such equations [7] in combination with the method of successive approximations the following statement on the local solvability of problem (3.14), (3.10) for $\chi > 0$ can be proved.

Let us assume that $H_0(\theta)$ is a positive function, periodic in θ with the period 2π, and belongs to the Hölder class $C^{4+\alpha}$, $0 < \alpha < 1$. Then there exists a $T > 0$ such that problem (3.14), (3.10) has a unique solution on the interval $0 \le t \le T$ in the class of positive functions which are periodic in θ with the period 2π. This solution is infinitely differentiable for $0 < t \le T$.

Since H is proportional to the film thickness, only nonnegative solutions of problem (3.14), (3.10) have a physical meaning. The fact that H is positive means that the film covers the surface of the cylinder completely. It would be interesting to prove the existence of a solution of problem (3.14), (3.10) in the large in the class of nonnegative functions and to find out whether the solution of this problem with $H_0 > 0$ vanishes for $t > 0$.

If $\chi = 0$, Eq. (3.14) is transformed into the first-order equation

$$H_t + \left(H - \frac{\mu}{3} H^3 \cos\theta\right)_\theta = 0, \tag{3.15}$$

which was also obtained and studied in [2]. When $\mu \neq 0$ the Cauchy problem (3.15), (3.10) has a smooth solution on a certain finite time interval for any smooth function H_0. In the process of evolution, however, discontinuities can be formed in the function H.

By determining the function $H(\theta, t)|_{\varkappa=0}$ as the solution of problem (3.14), (3.10) and substituting the result into Eqs. (3.12) and (3.13) we find the solution of problem (3.5)-(3.10) for $\varkappa = 0$. The function $V(y, \theta, t)|_{\varkappa=0}$ constructed in this way generally does not satisfy the initial condition (3.11). In order to eliminate the discrepancy we introduce the function $\xi(y, \theta, \tau, t)$ which satisfies the relations

$$\begin{gathered}\xi_\tau = \xi_{yy} \text{ for } 0 < y < H(\theta, t)|_{\varkappa=0},\\ \xi = 0 \text{ at } y = 0,\ \xi_y = 0 \text{ for } y = H(\theta, t)|_{\varkappa=0},\\ \xi = V_0(y, \theta) - V(y, \theta, 0)|_{\varkappa=0} \text{ at } \tau = 0.\end{gathered} \tag{3.16}$$

Here $\tau = \varkappa^{-1} t$ is the "fast time." We assume that $V_0 = 1$ at $y = 0$ and $V_{0,y} = 0$ for $y = H_0(\theta)$.

The solution of problem (3.16) and all its derivatives decrease exponentially as $\tau \to \infty$. Therefore, as $\varkappa \to 0$ the function $\xi(y, \theta, t/\varkappa, t)$ will be a boundary-layer type of function. A direct check shows that the functions H, U, and V determined by the equations

$$H = H(\theta, t)|_{\varkappa=0},\quad V = V(y, \theta, t)|_{\varkappa=0} + \xi(y, \theta, t/\varkappa, t),$$

$$U = U(y, \theta, t)|_{\varkappa=0} - \int_0^y \xi_\theta(x, \theta, t/\varkappa, t)\, dx,$$

satisfy Eqs. (3.6), (3.7), (3.10), and (3.11), and when they are substituted into Eqs. (3.5), (3.8), and (3.9) the value of the error as $\varkappa \to 0$ at any t, $0 < t \le T$, is of the order $O(\varkappa)$. This is the basis for calling the triplet of functions H, U, V the formal asymptotic solution of problem (3.5)-(3.11) as $\varkappa \to 0$. Functions H, U, V can be constructed which satisfy Eqs. (3.5)-(3.11) with an accuracy $0(\varkappa^n)$ as $\varkappa \to 0$, where n is any integer (this moment is not considered in this paper).

We note that for fixed values of Re and β the quantities $\varkappa$ and χ approach zero as $\varepsilon \to 0$. However, the limiting problem for the equations of a thin film when $\varkappa = \chi = 0$ has at least two defects: nonremovable discontinuities may appear in its solution; it is impossible to specify an arbitrary initial velocity field for the limiting equations.

Since χ is proportional to the surface tension, it follows from the preceding discussion that the effect of capillarity prevents the formation of shock waves in a thin film. The parabolic equation (3.14) is physically a natural regularization of the hyperbolic equation (3.15) to which the limiting problem reduces, and the parameter χ is the regularization parameter.

On the other hand, by taking account of the inertial term $\varkappa V_t$ in Eq. (3.5) problem (3.5)-(3.11) can be solved for arbitrary initial data. For small $\varkappa$, however, information on the initial velocity field, but not on the initial film profile, is rapidly forgotten in the motion process.

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TURBULENT VISCOSITY FOR INCOMPRESSIBLE GRADIENT FLOWS BEFORE SEPARATION AND ON A ROUGH SURFACE

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The existing finite-difference methods of turbulent boundary-layer calculation, where various modifications of turbulent viscosity (mixing path lengths) are used for closure of the system of equations, lead to great differences between the calculated and experimental data for highly nonequilibrium (close to separation) flows [1-3]. One of the probable reasons for the observed disagreement is that existing models of turbulent viscosity contain insufficient information about the previous history of the flow. In particular, the relation for turbulent viscosity in the external part of a boundary layer [2] or, for instance, the relationship used in [4]

$$\mu_T = \rho(\lambda y_e)^2|\partial u/\partial y| \quad (1)$$

in explicit form is quite independent of the previous history. The value of λ in (1) is constant and is usually taken as 0.09.

Correlation of the results of the experiments of Goldberg [3] and Schubauer and Spandenberg [1] showed that in the external part of the boundary layer the numerical value of λ can vary approximately from 0.045 to 0.090, i.e., $\lambda \neq$ const along the streamline. At the same time, as will be shown below, the value of λ can have a great effect on the fullness of the profile and the integral characteristics of the layer.

To determine the characteristics of the boundary layer before separation the authors of [4, 5] assumed various forms of dimensionless "universal" velocity profiles and obtained agreement with experiments by the introduction of empirical coefficients into the velocity profile relations.

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